On the Basis Property of the Root Functions of Some Class of Non-self-adjoint Sturm–Liouville Operators.

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Abstract

We obtain the asymptotic formulas for the eigenvalues and eigenfunctions of the Sturm-Liouville operators with some regular boundary conditions. Using these formulas, we find sufficient conditions on the potential q such that the root functions of these operators do not form a Riesz basis.

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1 Introduction and Preliminary Facts

Let T_1, T_2, T_3 and T_4 be the operators generated in $L_2[0,1]$ by the differential expression

$$l\,(y) = -y'' + q(x)y \tag{1}$$

and the following boundary conditions:

$$y_0' + \beta y_1' = 0,\ \ y_0 - y_1 = 0, \tag{2}$$

$$y_0' + \beta y_1' = 0,\ \ y_0 + y_1 = 0, \tag{3}$$

$$y_0' - y_1' = 0,\ \ y_0 + \alpha y_1 = 0, \tag{4}$$

and

$$y_0' + y_1' = 0,\ \ y_0 + \alpha y_1 = 0 \tag{5}$$

respectively, where $q(x)$ is a complex-valued summable function on $[0,1]$, $\beta \neq \pm 1$ and $\alpha \neq \pm 1$.

In conditions (2), (3), (4) and (5) if $\beta = 1$, $\beta = -1$, $\alpha = 1$ and $\alpha = -1$ respectively, then any $\lambda \in \mathbb{C}$ is an eigenvalue of infinite multiplicity. In (2) and (4) if $\beta = -1$ and $\alpha = -1$ then they are periodic boundary conditions; In (3) and (5) if $\beta = 1$ and $\alpha = 1$ then they are antiperiodic boundary conditions.

These boundary conditions are regular but not strongly regular. Note that, if the boundary conditions are strongly regular, then the root functions form a Riesz basis (this result was proved independently in [6], [10] and [17]). In the case when an operator is regular but not strongly regular, the root functions generally do not form even usual basis. However,

Shkalikov [20], [21] proved that they can be combined in pairs, so that the corresponding 2-dimensional subspaces form a Riesz basis of subspaces.

In the regular but not strongly regular boundary conditions, periodic and antiperiodic boundary conditions are the ones more commonly studied. Therefore, let us briefly describe some historical developments related to the Riesz basis property of the root functions of the periodic and antiperiodic boundary value problems. First results were obtained by Kerimov and Mamedov [8]. They established that, if

$$q \in C^4[0,1],\ q(1) \neq q(0),$$

then the root functions of the operator $L_0(q)$ form a Riesz basis in $L_2[0,1]$, where $L_0(q)$ denotes the operator generated by (1) and the periodic boundary conditions.

The first result in terms of the Fourier coefficients of the potential q was obtained by Dernek and Veliev [1]. They proved that if the conditions

$$\lim_{n\to\infty} \frac{\ln|n|}{nq_{2n}} = 0, \tag{6}$$

$$q_{2n} \sim q_{-2n} \tag{7}$$

hold, then the root functions of $L_0(q)$ form a Riesz basis in $L_2[0,1]$, where $q_n =: (q, e^{i2\pi nx})$ is the Fourier coefficient of q and everywhere, without loss of generality, it is assumed that $q_0 = 0$. Here $(.,.)$ denotes the inner product in $L_2[0,1]$ and $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$. Makin [11] improved this result. Using another method he proved that the assertion on the Riesz basis property remains valid if condition (7) holds, but condition (6) is replaced by a less restrictive one: $q \in W_1^s[0,1]$,

$$q^{(k)}(0) = q^{(k)}(1), \quad \forall\, k = 0, 1, ..., s-1$$

holds and $\mid q_{2n} \mid > cn^{-s-1}$ with some $c > 0$ for sufficiently large n, where s is a nonnegative integer. Besides, some conditions which imply the absence of the Riesz basis property were presented in [11]. Shkalilov and Veliev obtained in [22] more general results which cover all results discussed above.

The other interesting results about periodic and antiperiodic boundary conditions were obtained in [2-5, 7, 14-16, 24, 25].

The basis properties of regular but not strongly regular other some problems are studied in [9,12,13]. It was proved in [12] that the system of the root functions of the operator generated by (1) and the boundary conditions

$$y'(1) - (-1)^{\sigma} y'(0) + \gamma y(0) = 0$$
$$y(1) - (-1)^{\sigma} y(0) = 0,$$

forms an unconditional basis of the space $L_2[0,1]$, where $q(x)$ is an arbitrary complex-valued function from the class $L_1[0,1]$, γ is an arbitrary nonzero complex constant and $\sigma = 0, 1$. Kerimov and Kaya proved [9] that the system of the root functions of the spectral problem

$$y^{(4)} + p_2(x) y'' + p_1(x) y' + p_0(x) y = \lambda y,\ 0 < x < 1,$$
$$y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) + \sum_{l=0}^{s-1} \alpha_{s,l} y^{(l)}(0) = 0,\ s = 1, 2, 3,$$
$$y(1) - (-1)^{\sigma} y(0) = 0,$$

forms a basis in the space $L_p(0,1)$, $1 < p < \infty$, when $\alpha_{3,2} + \alpha_{1,0} \neq \alpha_{2,1}$, $p_j(x) \in W_1^j(0,1)$,

$j=1,2$, and $p_0(x)\in L_1(0,1)$; moreover, this basis is unconditional for $p=2$, where λ is a spectral parameter; $p_j(x)\in L_1(0,1)$, $j=1,2,3$, are complex-valued functions; $\alpha_{s,l}$, $s=1,2,3$, $l=\overline{0,s-1}$ are arbitrary complex constants; and $\sigma=0,1$.

It was shown in [19] that if

$$q(x)=q(1-x)\,,\ \forall x\in[0,1]\,,$$

then the spectrum of each of the problems T_1, and T_3, coincides with the spectrum of the periodic problem and the spectrum of each of the problems T_2, and T_4, coincides with the spectrum of the antiperiodic problem.

In this paper we prove that if

$$\lim_{n\to\infty}\frac{\ln|n|}{ns_{2n}}=0, \tag{8}$$

where $s_k=(q,\sin 2\pi kx)$, then the large eigenvalues of the operators T_1 and T_3 are simple. Moreover, if there exists a sequence $\{n_k\}$ such that (8) holds when n is replaced by n_k, then the root functions of these operators do not form a Riesz basis.

Similarly, if

$$\lim_{n\to\infty}\frac{\ln|n|}{ns_{2n+1}}=0, \tag{9}$$

then the large eigenvalues of the operators T_2 and T_4 are simple and if there exists a sequence $\{n_k\}$ such that (9) holds when n is replaced by n_k, then the root functions of these operators do not form a Riesz basis.

Moreover we obtain asymptotic formulas of arbitrary order for the eigenvalues and eigenfunctions of the operators T_1,T_2,T_3 and T_4.

2 Main Results

We will focus only on the operator T_1. The investigations of the operators T_2,T_3 and T_4 are similar. It is well-known that (see formulas (47a), (47b)) in page 65 of [18]) the eigenvalues of the operators $T_1(q)$ consist of the sequences $\{\lambda_{n,1}\},\{\lambda_{n,2}\}$ satisfying

$$\lambda_{n,j}=(2n\pi)^2+O(n^{1/2}) \tag{10}$$

for $j=1,2$. From this formula one can easily obtain the following inequality

$$\left|\lambda_{n,j}-(2\pi k)^2\right|=|2(n-k)\pi|\,|2(n+k)\pi|+O(n^{\frac{1}{2}})>n \tag{11}$$

for $j=1,2$; $k\neq n$; $k=0,1,...$; and $n\geq N$, where we denote by N a sufficiently large positive integer, that is, $N\gg 1$.

It is easy to verify that if $q(x)=0$ then the eigenvalues of the operator T_1, denoted by $T_1(0)$, are $\lambda_n=(2\pi n)^2$ for $n=0,1,\ldots$ The eigenvalue 0 is simple and the corresponding eigenfunction is 1. The eigenvalues $\lambda_n=(2\pi n)^2$ for $n=1,2,\ldots$ are double and the corresponding eigenfunctions and associated functions are

$$y_n(x)=\cos 2\pi nx \quad \& \quad \phi_n(x)=\left(\frac{\beta}{1+\beta}-x\right)\frac{\sin 2\pi nx}{4\pi n} \tag{12}$$

respectively. Note that for any constant c, $\phi_n(x)+cy_n(x)$ is also an associated function. It can be shown that the adjoint operator $T_1^*(0)$ is associated with the boundary conditions:

$$y_1+\overline{\beta}y_0=0,\ y_1'-y_0'=0. \tag{13}$$

It is easy to see that, 0 is a simple eigenvalue of $T_1^*(0)$ and the corresponding eigenfunction is $y_0^*(x) = x - \dfrac{1}{1+\overline{\beta}}$. The other eigenvalues $\lambda_n^* = (2\pi n)^2$ for $n = 1, 2, \ldots$, are double and the corresponding eigenfunctions and associated functions are

$$y_n^*(x) = \sin 2\pi nx \quad \& \quad \phi_n^*(x) = \left(x - \frac{1}{1+\overline{\beta}}\right)\frac{\cos 2\pi nx}{4\pi n}$$

respectively.

Let

$$\varphi_n(x) := \frac{16\pi n(\beta+1)}{\beta-1}\phi_n(x) = \frac{4(\beta+1)}{\beta-1}\left(\frac{\beta}{1+\beta} - x\right)\sin 2\pi nx \tag{14}$$

and

$$\varphi_n^*(x) := \frac{16\pi n\left(\overline{\beta}+1\right)}{\overline{\beta}-1}\phi_n^*(x) = \frac{4\left(\overline{\beta}+1\right)}{\overline{\beta}-1}\left(x - \frac{1}{1+\overline{\beta}}\right)\cos 2\pi nx. \tag{15}$$

The system of the root functions of $T_1^*(0)$ can be written as $\{f_n : n \in \mathbb{Z}\}$, where

$$f_{-n} = \sin 2\pi nx, \ \forall n > 0 \ \ \& \ \ f_n = \varphi_n^*(x), \ \forall n \geq 0. \tag{16}$$

One can easily verify that it forms a basis in $L_2[0,1]$ and the biorthogonal system $\{g_n : n \in \mathbb{Z}\}$ is the system of the root functions of $T_1(0)$, where

$$g_{-n} = \varphi_n(x), \forall n > 0 \quad \& \quad g_n = \cos 2\pi nx, \forall n \geq 0, \tag{17}$$

since $(f_n, g_m) = \delta_{n,m}$.

To obtain the asymptotic formulas for the eigenvalues $\lambda_{n,j}$ and the corresponding normalized eigenfunctions $\Psi_{n,j}(x)$ of $T_1(q)$ we use (11) and the well-known relations

$$(\lambda_{N,j} - (2\pi n)^2)(\Psi_{N,j}, \sin 2\pi nx) = (q\Psi_{N,j}, \sin 2\pi nx) \tag{18}$$

and

$$\left(\lambda_{N,j} - (2\pi n)^2\right)(\Psi_{N,j}, \varphi_n^*) - \gamma_1 n(\Psi_{N,j}, \sin 2\pi nx) = (q\Psi_{N,j}, \varphi_n^*), \tag{19}$$

where

$$\gamma_1 = \frac{16\pi(\beta+1)}{\beta-1},$$

which can be obtained by multiplying both sides of the equality

$$-(\Psi_{N,j})'' + q(x)\Psi_{N,j} = \lambda_{N,j}\Psi_{N,j}$$

by $\sin 2\pi nx$ and φ_n^* respectively. It follows from (18) and (19) that

$$(\Psi_{N,j}, \sin 2\pi nx) = \frac{(q(x)\Psi_{N,j}, \sin 2\pi nx)}{\lambda_{N,j} - (2\pi n)^2}; \ N \neq n, \tag{20}$$

$$(\Psi_{N,j}, \varphi_n^*) = \frac{\gamma_1 n(q(x)\Psi_{N,j}, \sin 2\pi nx)}{\left(\lambda_{N,j} - (2\pi n)^2\right)^2} + \frac{(q(x)\Psi_{N,j}, \varphi_n^*)}{\lambda_{N,j} - (2\pi n)^2}; \ N \neq n. \tag{21}$$

Moreover, we use the following relations

$$(\Psi_{N,j},\overline{q}\sin 2\pi nx)=\sum_{n_1=0}^{\infty}[(q\varphi_{n_1},\sin 2\pi nx)\left(\Psi_{N,j},\sin 2\pi n_1x\right)+ \tag{22}$$
$$+\left(q\cos 2\pi n_1x,\sin 2\pi nx\right)\left(\Psi_{N,j},\varphi_{n_1}^{*}\right)],$$

$$(\Psi_{N,j},\overline{q}\varphi_{n}^{*})=\sum_{n_1=0}^{\infty}\left[(q\varphi_{n_1},\varphi_{n}^{*})\left(\Psi_{N,j},\sin 2\pi n_1x\right)+\left(q\cos 2\pi n_1x,\varphi_{n}^{*}\right)\left(\Psi_{N,j},\varphi_{n_1}^{*}\right)\right], \tag{23}$$

$$|(q\Psi_{N,j},\sin 2\pi nx)|<4M, \tag{24}$$
$$|(q\Psi_{N,j},\varphi_{n}^{*})|<4M, \tag{25}$$

for $N\gg 1$,where $M=\sup|q_n|$. These relations are obvious for $q\in L_2(0,1)$, since to obtain (22) and (23) we can use the decomposition of $\overline{q}\sin 2\pi nx$ and $\overline{q}\varphi_{n}^{*}$ by basis (16). For $q\in L_1(0,1)$ see Lemma 1 of [23].

To obtain the asymptotic formulas for the eigenvalues and eigenfunctions we iterate (18) and (19) by using (22), (23). First let us prove the following obvious asymptotic formulas for the eigenfunctions $\Psi_{n,j}$. The expansion of $\Psi_{n,j}$ by basis (17) can be written in the form

$$\Psi_{n,j}=u_{n,j}\varphi_{n}\left(x\right)+v_{n,j}\cos 2\pi nx+h_{n,j}\left(x\right), \tag{26}$$

where

$$u_{n,j}=\left(\Psi_{n,j},\sin 2\pi nx\right),\ v_{n,j}=\left(\Psi_{n,j},\varphi_{n}^{*}\right), \tag{27}$$

$$h_{n,j}\left(x\right)=\sum_{\substack{k=0\\k\neq n}}^{\infty}\left[\left(\Psi_{n,j},\sin 2\pi kx\right)\varphi_{k}\left(x\right)+\left(\Psi_{n,j},\varphi_{k}^{*}\right)\cos 2\pi kx\right].$$

Using (20), (21), (24) and (25) one can readily see that, there exists a constant C such that

$$\sup|h_{n,j}\left(x\right)|\leq C\left(\sum_{k\neq n}\left(\frac{1}{|\ \lambda_{n,j}-(2\pi k)^{2}\ |}+\frac{n}{\left|\left(\lambda_{n,j}-(2\pi k)^{2}\right)^{2}\right|}\right)\right)=O\left(\frac{\ln n}{n}\right) \tag{28}$$

and by (26) we get

$$\Psi_{n,j}=u_{n,j}\varphi_{n}\left(x\right)+v_{n,j}\cos 2\pi nx+O\left(\frac{\ln n}{n}\right). \tag{29}$$

Since $\Psi_{n,j}$ is normalized, we have

$$1=\left\|\Psi_{n,j}\right\|^{2}=\left(\Psi_{n,j},\Psi_{n,j}\right)=\left|u_{n,j}\right|^{2}\left\|\varphi_{n}\left(x\right)\right\|^{2}+\left|v_{n,j}\right|^{2}\left\|\cos 2\pi nx\right\|^{2}+$$
$$+u_{n,j}\overline{v_{n,j}}\left(\varphi_{n}\left(x\right),\cos 2\pi nx\right)+v_{n,j}\overline{u_{n,j}}\left(\cos 2\pi nx,\varphi_{n}\left(x\right)\right)+O\left(\frac{\ln n}{n}\right)$$
$$=\left(\frac{8}{3}\frac{|\beta|^{2}-\operatorname{Re}\beta+1}{|\beta-1|^{2}}\right)\left|u_{n,j}\right|^{2}+\frac{1}{2}\left|v_{n,j}\right|^{2}+O\left(\frac{\ln n}{n}\right),$$

that is,

$$a\left|u_{n,j}\right|^{2}+\frac{1}{2}\left|v_{n,j}\right|^{2}=1+O\left(\frac{\ln n}{n}\right), \tag{30}$$

where

$$a=\frac{8}{3}\frac{\left|\beta\right|^{2}-\operatorname{Re}\beta+1}{\left|\beta-1\right|^{2}}.$$

Note that $a\neq 0$, since $\left|\beta\right|^{2}+1>\left|\beta\right|.$

Now let us iterate (18). Using (22) in (18) we get

$$\left(\lambda_{n,j}-(2\pi n)^{2}\right)\left(\Psi_{n,j},\sin 2\pi nx\right)=$$

$$=\sum_{n_1=0}^{\infty}\left[\left(q\varphi_{n_1},\sin 2\pi nx\right)\left(\Psi_{n,j},\sin 2\pi n_1x\right)+\left(q\cos 2\pi n_1x,\sin 2\pi nx\right)\left(\Psi_{n,j},\varphi_{n_1}^{*}(x)\right)\right].$$

Isolating the terms in the right-hand side of this equality containing the multiplicands $\left(\Psi_{n,j},\sin 2\pi nx\right)$ and $\left(\Psi_{n,j},\varphi_{n}^{*}(x)\right)$ (i.e., case $n_1=n$), using (20) and (21) for the terms $\left(\Psi_{n,j},\sin 2\pi n_1x\right)$ and $\left(\Psi_{n,j},\varphi_{n_1}^{*}(x)\right)$ respectively (in the case $n_1\neq n$) we obtain

$$\left[\lambda_{n,j}-(2\pi n)^{2}-\left(q\varphi_{n},\sin 2\pi nx\right)\right]\left(\Psi_{n,j},\sin 2\pi nx\right)-\left(q\cos 2\pi nx,\sin 2\pi nx\right)\left(\Psi_{n,j},\varphi_{n}^{*}\right)=$$

$$=\sum_{\substack{n_1=0\\ n_1\neq n}}^{\infty}\left[\left(q\varphi_{n_1},\sin 2\pi nx\right)\left(\Psi_{n,j},\sin 2\pi n_1x\right)+\left(q\cos 2\pi n_1x,\sin 2\pi nx\right)\left(\Psi_{n,j},\varphi_{n_1}^{*}(x)\right)\right]$$

$$=\sum_{n_1}\left[a_1\left(\lambda_{n,j}\right)\left(q(x)\Psi_{n,j},\sin 2\pi n_1x\right)+b_1\left(\lambda_{n,j}\right)\left(q(x)\Psi_{n,j},\varphi_{n_1}^{*}\right)\right].$$

where

$$a_1\left(\lambda_{n,j}\right)=\frac{\left(q\varphi_{n_1},\sin 2\pi nx\right)}{\lambda_{n,j}-\left(2\pi n_1\right)^{2}}+\frac{\gamma_1 n_1\left(q\cos 2\pi n_1x,\sin 2\pi nx\right)}{\left(\lambda_{n,j}-\left(2\pi n_1\right)^{2}\right)^{2}},$$

$$b_1\left(\lambda_{n,j}\right)=\frac{\left(q\cos 2\pi n_1x,\sin 2\pi nx\right)}{\lambda_{n,j}-\left(2\pi n_1\right)^{2}}.$$

Using (22) and (23) for the terms $\left(q\Psi_{n,j},\sin 2\pi n_1x\right)$ and $\left(q\Psi_{n,j},\varphi_{n_1}^{*}\right)$ of the last summation we obtain

$$\left[\lambda_{n,j}-(2\pi n)^{2}-\left(q\varphi_{n},\sin 2\pi nx\right)\right]\left(\Psi_{n,j},\sin 2\pi nx\right)-\left(q\cos 2\pi nx,\sin 2\pi nx\right)\left(\Psi_{n,j},\varphi_{n}^{*}\right)=$$

$$=\sum_{n_1}\left[a_1\left(\lambda_{n,j}\right)\left(q\Psi_{n,j},\sin 2\pi n_1x\right)+b_1\left(\lambda_{n,j}\right)\left(q\Psi_{n,j},\varphi_{n_1}^{*}\right)\right]=$$

$$=\sum_{n_1}a_1\left(\sum_{n_2=0}^{\infty}\left[\left(q\varphi_{n_2},\sin 2\pi n_1x\right)\left(\Psi_{n,j},\sin 2\pi n_2x\right)+\left(q\cos 2\pi n_2x,\sin 2\pi n_1x\right)\left(\Psi_{n,j},\varphi_{n_2}^{*}(x)\right)\right]\right)+$$

$$+\sum_{n_1}b_1\left(\sum_{n_2=0}^{\infty}\left[\left(q\varphi_{n_2},\varphi_{n_1}^{*}\right)\left(\Psi_{n,j},\sin 2\pi n_2x\right)+\left(q\cos 2\pi n_2x,\varphi_{n_1}^{*}\right)\left(\Psi_{n,j},\varphi_{n_2}^{*}(x)\right)\right]\right).$$

Now isolating the terms for $n_2 = n$ we get

$$\left[\lambda_{n,j} - (2\pi n)^2 - (q\varphi_n, \sin 2\pi nx)\right] (\Psi_{n,j}, \sin 2\pi nx) - (q\cos 2\pi nx, \sin 2\pi nx) (\Psi_{n,j}, \varphi_n^*) =$$

$$= \sum_{n_1} \left[a_1 (q\varphi_n, \sin 2\pi n_1 x) + b_1 \left(q\varphi_n, \varphi_{n_1}^*\right)\right] (\Psi_{n,j}, \sin 2\pi nx) +$$

$$+ \sum_{n_1} \left[a_1 (q\cos 2\pi nx, \sin 2\pi n_1 x) + b_1 \left(q\cos 2\pi nx, \varphi_{n_1}^*\right)\right] (\Psi_{n,j}, \varphi_n^*(x)) +$$

$$= \sum_{n_1,n_2} \left(\left[a_1 (q\varphi_{n_2}, \sin 2\pi n_1 x) + b_1 \left(q\varphi_{n_2}, \varphi_{n_1}^*\right)\right] (\Psi_{n,j}, \sin 2\pi n_2 x) +\right) +$$

$$+ \sum_{n_1,n_2} \left[a_1 (q\cos 2\pi n_2 x, \sin 2\pi n_1 x) + b_1 \left(q\cos 2\pi n_2 x, \varphi_{n_1}^*\right)\right] \left(\Psi_{n,j}, \varphi_{n_2}^*\right).$$

Here and further the summations are taken under the conditions $n_i \neq n$ and $n_i = 0, 1, ...$ for $i = 1, 2, ...$ Introduce the notations

$$C_1 =: a_1,\ M_1 =: b_1,$$
$$C_2 =: a_1 a_2 + b_1 A_2 = C_1 a_2 + M_1 A_2,\ M_2 =: a_1 b_2 + b_1 B_2 = C_1 b_2 + M_1 B_2,$$
$$C_{k+1} =: C_k a_{k+1} + M_k A_{k+1},\ M_{k+1} =: C_k b_{k+1} + M_k B_{k+1};\ k = 2, 3, \ldots,$$

where

$$a_{k+1} = a_{k+1}(\lambda_{n,j}) = \frac{\left(q\varphi_{n_{k+1}}, \sin 2\pi n_k x\right)}{\lambda_{n,j} - (2\pi n_{k+1})^2} + \frac{\gamma_1 n_{k+1} (q\cos 2\pi n_{k+1} x, \sin 2\pi n_k x)}{\left(\lambda_{n,j} - (2\pi n_{k+1})^2\right)^2},$$

$$b_{k+1} = b_{k+1}(\lambda_{n,j}) = \frac{(q\cos 2\pi n_{k+1} x, \sin 2\pi n_k x)}{\lambda_{n,j} - (2\pi n_{k+1})^2},$$

$$A_{k+1} = A_{k+1}(\lambda_{n,j}) = \frac{\left(q\varphi_{n_{k+1}}, \varphi_{n_k}^*\right)}{\lambda_{n,j} - (2\pi n_{k+1})^2} + \frac{\gamma_1 n_{k+1} \left(q\cos 2\pi n_{k+1} x, \varphi_{n_k}^*\right)}{\left(\lambda_{n,j} - (2\pi n_{k+1})^2\right)^2},$$

$$B_{k+1} = B_{k+1}(\lambda_{n,j}) = \frac{\left(q\cos 2\pi n_{k+1} x, \varphi_{n_k}^*\right)}{\lambda_{n,j} - (2\pi n_{k+1})^2}.$$

Using these notations and repeating this iteration k times we get

$$\left[\lambda_{n,j} - (2\pi n)^2 - (q\varphi_n, \sin 2\pi nx) - \widetilde{A}_k(\lambda_{n,j})\right] (\Psi_{n,j}, \sin 2\pi nx) =$$
$$= \left[(q\cos 2\pi nx, \sin 2\pi nx) + \widetilde{B}_k(\lambda_{n,j})\right] (\Psi_{n,j}, \varphi_n^*(x)) + R_k, \tag{31}$$

where

$$\widetilde{A}_k(\lambda_{n,j}) = \sum_{m=1}^{k} \alpha_m(\lambda_{n,j}), \ \widetilde{B}_k(\lambda_{n,j}) = \sum_{m=1}^{k} \beta_m(\lambda_{n,j}),$$
$$\alpha_k(\lambda_{n,j}) = \sum_{n_1,\ldots,n_k} \left[C_k(q\varphi_n, \sin 2\pi n_k x) + M_k\left(q\varphi_n, \varphi^*_{n_k}\right)\right],$$
$$\beta_k(\lambda_{n,j}) = \sum_{n_1,\ldots,n_k} \left[C_k(q\cos 2\pi n x, \sin 2\pi n_k x) + M_k\left(q\cos 2\pi n x, \varphi^*_{n_k}\right)\right],$$
$$R_k = \sum_{n_1,\ldots,n_{k+1}} \left\{C_{k+1}(q\Psi_{n,j}, \sin 2\pi n_{k+1} x) + M_{k+1}\left(q\Psi_{n,j}, \varphi^*_{n_{k+1}}\right)\right\}.$$

It follows from (11), (24) and (25) that

$$\alpha_k(\lambda_{n,j}) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right), \beta_k(\lambda_{n,j}) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right), R_k = O\left(\left(\frac{\ln|n|}{n}\right)^{k+1}\right). \quad (32)$$

Therefore if we take limit in (31) for $k \to \infty$, we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - Q_n - A(\lambda_{n,j})\right] u_{n,j} = \left[P_n + B(\lambda_{n,j})\right] v_{n,j},$$

where

$$P_n = (q\cos 2\pi n x, \sin 2\pi n x), \ Q_n = (q\varphi_n, \sin 2\pi n x), \quad (33)$$

$$A(\lambda_{n,j}) = \sum_{m=1}^{\infty} \alpha_m(\lambda_{n,j}) = O\left(\frac{\ln|n|}{n}\right), \ B(\lambda_{n,j}) = \sum_{m=1}^{\infty} \beta_m(\lambda_{n,j}) = O\left(\frac{\ln|n|}{n}\right). \quad (34)$$

Thus iterating (18) we obtained (31). Now starting to iteration from (19) instead of (18) and using (23), (22) and arguing as in the previous iteration, we get

$$\left[\lambda_{n,j} - (2\pi n)^2 - P^*_n - A'_k(\lambda_{n,j})\right] v_{n,j} = \left[\gamma_1 n + Q^*_n + B'_k(\lambda_{n,j})\right] u_{n,j} + R'_k, \quad (35)$$

where

$$P^*_n = (q\cos 2\pi n x, \varphi^*_n), \ Q^*_n = (q\varphi_n, \varphi^*_n), \quad (36)$$

$$A'_k(\lambda_{n,j}) = \sum_{m=1}^{k} \alpha'_m(\lambda_{n,j}), \ B'_k(\lambda_{n,j}) = \sum_{m=1}^{k} \beta'_m(\lambda_{n,j}),$$
$$\alpha'_k(\lambda_{n,j}) = \sum_{n_1,\ldots,n_k} \left[\widetilde{C}_k(q\cos 2\pi n x, \sin 2\pi n_k x) + \widetilde{M}_k\left(q\cos 2\pi n x, \varphi^*_{n_k}\right)\right],$$
$$\beta'_k(\lambda_{n,j}) = \sum_{n_1,\ldots,n_k} \left[\widetilde{C}_k(q\varphi_n, \sin 2\pi n_k x) + \widetilde{M}_k\left(q\varphi_n, \varphi^*_{n_k}\right)\right],$$
$$R'_k = \sum_{n_1,\ldots,n_{k+1}} \left\{\widetilde{C}_{k+1}(q\Psi_{n,j}, \sin 2\pi n_{k+1} x) + \widetilde{M}_{k+1}\left(q\Psi_{n,j}, \varphi^*_{n_{k+1}}\right)\right\},$$

$$\widetilde{C}_{k+1} = \widetilde{C}_k a_{k+1} + \widetilde{M}_k A_{k+1}, \ \widetilde{M}_{k+1} = \widetilde{C}_k b_{k+1} + \widetilde{M}_k B_{k+1}; \ k = 0, 1, 2, \ldots,$$

$$\widetilde{C}_1 = A_1\left(\lambda_{n,j}\right) = \frac{\left(q\varphi_{n_1}, \varphi_n^*\right)}{\lambda_{n,j} - \left(2\pi n_1\right)^2} + \frac{\gamma_1 n_1 \left(q \cos 2\pi n_1 x, \varphi_n^*\right)}{\left(\lambda_{n,j} - \left(2\pi n_1\right)^2\right)^2},$$

$$\widetilde{M}_1 = B_1\left(\lambda_{n,j}\right) = \frac{\left(q \cos 2\pi n_1 x, \varphi_n^*\right)}{\lambda_{n,j} - \left(2\pi n_1\right)^2}.$$

Similar to (32) one can verify that

$$\alpha_k'\left(\lambda_{n,j}\right) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right), \beta_k'\left(\lambda_{n,j}\right) = O\left(\left(\frac{\ln|n|}{n}\right)^k\right), R_k' = O\left(\left(\frac{\ln|n|}{n}\right)^{k+1}\right). \quad (37)$$

If we take limit in (35) for $k \to \infty$, we obtain

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'\left(\lambda_{n,j}\right)\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + B'\left(\lambda_{n,j}\right)\right] u_{n,j},$$

where

$$A'\left(\lambda_{n,j}\right) = \sum_{m=1}^{\infty} \alpha_m'\left(\lambda_{n,j}\right) = O\left(\frac{\ln|n|}{n}\right), \; B'\left(\lambda_{n,j}\right) = \sum_{m=1}^{\infty} \beta_m'\left(\lambda_{n,j}\right) = O\left(\frac{\ln|n|}{n}\right). \quad (38)$$

To get the main results of this paper we use the following system of equations, obtained above, with respect to $u_{n,j}$ and $v_{n,j}$

$$\left[\lambda_{n,j} - (2\pi n)^2 - Q_n - A\left(\lambda_{n,j}\right)\right] u_{n,j} = \left[P_n + B\left(\lambda_{n,j}\right)\right] v_{n,j}, \quad (39)$$

$$\left[\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'\left(\lambda_{n,j}\right)\right] v_{n,j} = \left[\gamma_1 n + Q_n^* + B'\left(\lambda_{n,j}\right)\right] u_{n,j}, \quad (40)$$

where

$$Q_n = \left(q\varphi_n, \sin 2\pi n x\right) =$$

$$= -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}\left(xq(x), \cos 4\pi n x\right) - \frac{2\beta}{\beta-1}\left(q(x), \cos 4\pi n x\right) \quad (41)$$

$$= -\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + o(1), \quad (42)$$

$$P_n^* = \left(q \cos 2\pi n x, \varphi_n^*\right) =$$

$$= \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + \frac{2(\beta+1)}{\beta-1}\left(xq(x), \cos 4\pi n x\right) - \frac{2}{\beta-1}\left(q(x), \cos 4\pi n x\right) \quad (43)$$

$$= \frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx + o(1), \quad (44)$$

$$P_n = \left(q \cos 2\pi n x, \sin 2\pi n x\right) = \frac{1}{2}\left(q, \sin 4\pi n x\right) = o(1), \quad (45)$$

$$Q_n^* = \left(q\varphi_n, \varphi_n^*\right) = 8\left(\frac{\beta_1+1}{\beta_1-1}\right)^2 \int_0^1 q(x)\left(\frac{\beta_1}{1+\beta_1} - x\right)\left(x - \frac{1}{1+\beta_1}\right)\sin 4\pi n x dx = o(1). \quad (46)$$

Note that (39), (40) with (34), (38) give us

$$\left[\lambda_{n,j}-(2\pi n)^2-Q_n+O\left(\frac{\ln|n|}{n}\right)\right]u_{n,j}=\left[P_n+O\left(\frac{\ln|n|}{n}\right)\right]v_{n,j}, \tag{47}$$

$$\left[\lambda_{n,j}-(2\pi n)^2-P_n^*+O\left(\frac{\ln|n|}{n}\right)\right]v_{n,j}=\left[\gamma_1 n+Q_n^*+O\left(\frac{\ln|n|}{n}\right)\right]u_{n,j}. \tag{48}$$

Introduce the notations

$$\begin{aligned} c_n&=(q,\cos 2\pi nx)\,,\ s_n=(q,\sin 2\pi nx)\\ c_{n,1}&=(xq,\cos 2\pi nx)\,,\ s_{n,1}=(xq,\sin 2\pi nx)\\ c_{n,2}&=\left(x^2q,\cos 2\pi nx\right),\ s_{n,2}=\left(x^2q,\sin 2\pi nx\right). \end{aligned} \tag{49}$$

In these notations we have

$$Q_n=-\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx+\frac{2(\beta+1)}{\beta-1}c_{2n,1}-\frac{2\beta}{\beta-1}c_{2n} \tag{50}$$

$$P_n^*=\frac{2(\beta+1)}{\beta-1}\int_0^1 xq(x)\,dx+\frac{2(\beta+1)}{\beta-1}c_{2n,1}-\frac{2}{\beta-1}c_{2n} \tag{51}$$

$$P_n=\frac{1}{2}s_{2n} \tag{52}$$

$$Q_n^*=-8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,2}+8\left(\frac{\beta+1}{\beta-1}\right)^2 s_{2n,1}-\frac{8\beta}{(\beta-1)^2}s_{2n}. \tag{53}$$

Theorem 1 *For $j=1,2$ the following statements hold:*

(a) Any eigenfunction $\Psi_{n,j}$ of T_1 corresponding to the eigenvalue $\lambda_{n,j}$ defined in (10) satisfies

$$\Psi_{n,j}=\sqrt{2}\cos 2\pi nx+O\left(n^{-1/2}\right). \tag{54}$$

Moreover there exists N such that for all $n>N$ the geometric multiplicity of the eigenvalue $\lambda_{n,j}$ is 1.

(b) A complex number $\lambda\in U(n)=:\{\lambda:\left|\lambda-(2\pi n)^2\right|\le n\}$ is an eigenvalue of T_1 if and only if it is a root of the equation

$$\begin{aligned}&\left[\lambda-(2\pi n)^2-Q_n-A(\lambda)\right]\left[\lambda-(2\pi n)^2-P_n^*-A'(\lambda)\right]-\\ &\qquad-\left[P_n+B(\lambda)\right]\left[\gamma_1 n+Q_n^*+B'(\lambda)\right]=0.\end{aligned} \tag{55}$$

Moreover $\lambda\in U(n)$ is a double eigenvalue of T_1 if and only if it is a double root of (55) .

Proof. (a) By (10) the left hand side of (48) is $O(n^{1/2})$, which implies that $u_{n,j}=O(n^{-1/2})$. Therefore from (29) we obtain (54). Now suppose that there are two linearly independent eigenfunctions corresponding to $\lambda_{n,j}$. Then there exists an eigenfunction satisfying

$$\Psi_{n,j}=\sqrt{2}\sin 2\pi nx+o(1)$$

which contradicts (54).

(b) First we prove that the large eigenvalues $\lambda_{n,j}$ are the roots of the equation (55). It follows from (54), (27) and (15) that $v_{n,j}\neq 0$. If $u_{n,j}\neq 0$ then multiplying the equations (39) and (40) side by side and then canceling $v_{n,j}u_{n,j}$ we obtain (55) . If $u_{n,j}=0$ then by

(39) and (40) we have $P_n + B(\lambda_{n,j}) = 0$ and $\lambda_{n,j} - (2\pi n)^2 - P_n^* - A'(\lambda_{n,j}) = 0$ which mean that (55) holds. Thus in any case $\lambda_{n,j}$ is a root of (55).

Now we prove that the roots of (55) lying in $U(n)$ are the eigenvalues of T_1. Let $F(\lambda)$ be the left-hand side of (55) which can be written as

$$F(\lambda) = (\lambda - (2\pi n)^2)^2 - (Q_n + A(\lambda) + P_n^* + A'(\lambda)) \left(\lambda - (2\pi n)^2\right) + \\ + (Q_n + A(\lambda))(P_n^* + A'(\lambda)) - (P_n + B(\lambda))(\gamma_1 n + Q_n^* + B'(\lambda)) \quad (56)$$

and

$$G(\lambda) = (\lambda - (2\pi n)^2)^2.$$

One can easily verify that the inequality

$$\mid F(\lambda) - G(\lambda) \mid < \mid G(\lambda) \mid \quad (57)$$

holds for all λ from the boundary of $U(n)$. Since the function $G(\lambda)$ has two roots in the set $U(n)$, by the Rouche's theorem we obtain that $F(\lambda)$ has two roots in the same set. Thus T_1 has two eigenvalues (counting with multiplicities) lying in $U(n)$ that are the roots of (55). On the other hand, (55) has preciously two roots (counting with multiplicities) in $U(n)$. Therefore $\lambda \in U(n)$ is an eigenvalue of T_1 if and only if (55) holds.

If $\lambda \in U(n)$ is a double eigenvalue of T_1 then it has no other eigenvalues in $U(n)$ and hence (55) has no other roots. This implies that λ is a double root of (55). By the same way one can prove that if λ is a double root of (55) then it is a double eigenvalue of T_1. ■

Let us consider (55) in detail. If we substitute $t =: \lambda - (2\pi n)^2$ then it becomes

$$t^2 - (Q_n + A(\lambda) + P_n^* + A'(\lambda))\, t + \\ + (Q_n + A(\lambda))(P_n^* + A'(\lambda)) - (P_n + B(\lambda))(\gamma_1 n + Q_n^* + B'(\lambda)) = 0. \quad (58)$$

The solutions of this equation are

$$t_{1,2} = \frac{(Q_n + P_n^* + A + A') \pm \sqrt{\Delta(\lambda)}}{2},$$

where

$$\Delta(\lambda) = (Q_n + P_n^* + A + A')^2 - 4(Q_n + A)(P_n^* + A') + 4(P_n + B)(\gamma_1 n + Q_n^* + B')$$

which can be written in the form

$$\Delta(\lambda) = (Q_n - P_n^* + A - A')^2 + 4(P_n + B)(\gamma_1 n + Q_n^* + B'). \quad (59)$$

Clearly the eigenvalue $\lambda_{n,j}$ is a root either of the equation

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[(Q_n + P_n^* + A + A') - \sqrt{\Delta(\lambda)}\right] \quad (60)$$

or of the equation

$$\lambda = (2\pi n)^2 + \frac{1}{2}\left[(Q_n + P_n^* + A + A') + \sqrt{\Delta(\lambda)}\right]. \quad (61)$$

Now let us examine $\Delta(\lambda_{n,j})$ in detail. If (8) holds then one can readily see from (34), (38), (50), (51) and (59) that

$$\Delta(\lambda_{n,j}) = 2\gamma_1 n s_{2n}(1 + o(1)). \quad (62)$$

Taking into account the last three equality and (34), (38), (50), (51), we see that (60) and (61) have the form

$$\lambda=(2\pi n)^{2}-\frac{\sqrt{2\gamma_{1}}}{2}\sqrt{ns_{2n}}(1+o(1)), \tag{63}$$

$$\lambda=(2\pi n)^{2}+\frac{\sqrt{2\gamma_{1}}}{2}\sqrt{ns_{2n}}(1+o(1)). \tag{64}$$

Theorem 2 *If (8) holds, then the large eigenvalues $\lambda_{n,j}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j}=(2\pi n)^{2}+(-1)^{j}\,\frac{\sqrt{2\gamma_{1}}}{2}\sqrt{ns_{2n}}(1+o(1)). \tag{65}$$

for $j=1,2$. Moreover, if there exists a sequence $\{n_k\}$ such that (8) holds when n is replaced by n_k, then the root functions of T_1 do not form a Riesz basis.

Proof. To prove that the large eigenvalues $\lambda_{n,j}$ are simple let us show that one of the eigenvalues, say $\lambda_{n,1}$ satisfies (65) for $j=1$ and the other $\lambda_{n,2}$ satisfies (65) for $j=2$. Let us prove that each of the equations (60) and (61) has a unique root in $U(n)$ by proving that

$$(2\pi n)^{2}+\frac{1}{2}\left[(Q_n+P_n^{*}+A+A')\pm\sqrt{\Delta\left(\lambda\right)}\right]$$

is a contraction mapping. For this we show that there exist positive real numbers K_1, K_2, K_3 such that

$$\mid A\left(\lambda\right)-A(\mu)\mid<K_{1}\mid\lambda-\mu\mid,\quad\mid A'(\lambda)-A'(\mu)\mid<K_{2}\mid\lambda-\mu\mid, \tag{66}$$

$$\left|\sqrt{\Delta\left(\lambda\right)}-\sqrt{\Delta\left(\mu\right)}\right|<K_{3}\mid\lambda-\mu\mid, \tag{67}$$

where $K_1+K_2+K_3<1$. The proof of (66) is similar to the proof of (56) of the paper [26].

Now let us prove (67). By (62) and (8) we have

$$\left(\sqrt{\Delta\left(\lambda\right)}\right)^{-1}=o(1).$$

On the other hand arguing as in the proof of (56) of the paper [26] we get

$$\frac{d}{d\lambda}\Delta\left(\lambda\right)=O(1).$$

Hence in any case we have

$$\frac{d}{d\lambda}\sqrt{\Delta\left(\lambda\right)}=\frac{\frac{d}{d\lambda}\Delta\left(\lambda\right)}{2\sqrt{\Delta\left(\lambda\right)}}=o(1).$$

Thus by the fixed point theorem, each of the equations (60) and (61) has a unique root λ_1 and λ_2 respectively. Clearly by (63) and (64), we have $\lambda_1\neq\lambda_2$ which implies that the equation (55) has two simple root in $U\left(n\right)$. Therefore by Theorem 1(b), λ_1 and λ_2 are the eigenvalues of T_1 lying in $U\left(n\right)$, that is, they are $\lambda_{n,1}$ and $\lambda_{n,2}$, which proves the simplicity of the large eigenvalues and the validity of (65).

If there exists a sequence $\{n_k\}$ such that (8) holds when n is replaced by n_k, then by Theorem 1(a)

$$(\Psi_{n_k,1},\Psi_{n_k,2})=1+O\left(n_k^{-1/2}\right).$$

Now it follows from the theorems of [20,21] (see also Lemma 3 of [24]) that the root functions of T_1 do not form a Riesz basis. ■

Now let us consider the operators T_2, T_3 and T_4. First we consider the operator T_3.

It is well-known that (see formulas (47a), (47b)) in page 65 of [18]) the eigenvalues of the operators $T_3(q)$ consist of the sequences $\{\lambda_{n,1,3}\}, \{\lambda_{n,2,3}\}$ satisfying (10) when $\lambda_{n,j}$ is replaced by $\lambda_{n,j,3}$. The eigenvalues, eigenfunctions and associated functions of T_3 are

$$\lambda_n = (2\pi n)^2 ; \; n = 0, 1, 2, \ldots$$
$$y_0(x) = x - \frac{\alpha}{1+\alpha}, \; y_n(x) = \sin 2\pi n x; \; n = 1, 2, \ldots$$
$$\phi_n(x) = \left(x - \frac{\alpha}{1+\alpha}\right) \frac{\cos 2\pi n x}{4\pi n}; \; n = 1, 2, \ldots.$$

respectively. The biorthogonal systems analogous to (16), (17) are

$$\left\{\cos 2\pi n x, \frac{4(1+\overline{\alpha})}{1-\overline{\alpha}} \left(\frac{1}{1+\overline{\alpha}} - x\right) \sin 2\pi n x\right\}_{n=0}^{\infty} \tag{68}$$

$$\left\{\sin 2\pi n x, \frac{4(1+\alpha)}{1-\alpha} \left(x - \frac{\alpha}{1+\alpha}\right) \cos 2\pi n x\right\}_{n=0}^{\infty} \tag{69}$$

respectively.

Analogous formulas to (18) and (19) are

$$\left(\lambda_{N,j} - (2\pi n)^2\right) (\Psi_{N,j}, \cos 2\pi n x) = (q(x)\Psi_{N,j}, \cos 2\pi n x) \tag{70}$$

$$\left(\lambda_{N,j} - (2\pi n)^2\right) (\Psi_{N,j}, \varphi_n^*) - \gamma_3 n (\Psi_{N,j}, \cos 2\pi n x) = (q(x)\Psi_{N,j}, \varphi_n^*) \tag{71}$$

respectively, where

$$\gamma_3 = \frac{16\pi(1+\alpha)}{1-\alpha}.$$

Instead of (16)-(19) using (68)-(71) and arguing as in the proofs of Theorem 1 and Theorem 2 we obtain the following results for T_3.

Theorem 3 *If (8) holds, then the large eigenvalues $\lambda_{n,j,3}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j,3} = (2\pi n)^2 + (-1)^j \frac{\sqrt{2\gamma_3}}{2} \sqrt{n s_{2n}} (1 + o(1)). \tag{72}$$

for $j = 1, 2$. The eigenfunctions $\Psi_{n,j,3}$ corresponding to $\lambda_{n,j,3}$ obey

$$\Psi_{n,j,3} = \sqrt{2} \sin 2\pi n x + O\left(n^{-1/2}\right). \tag{73}$$

Moreover, if there exists a sequence $\{n_k\}$ such that (8) holds when n is replaced by n_k, then the root functions of T_3 do not form a Riesz basis.

Now let us consider the operator T_2. It is well-known that (see formulas (47a), (47b)) in page 65 of [18]) the eigenvalues of the operators $T_2(q)$ consist of the sequences $\{\lambda_{n,1,2}\}, \{\lambda_{n,2,2}\}$ satisfying

$$\lambda_{n,j,2} = (2n\pi + \pi)^2 + O(n^{1/2}), \tag{74}$$

for $j=1,2$. The eigenvalues, eigenfunctions and associated functions of T_2 are

$$\lambda_n = (\pi + 2\pi n)^2\,,\ y_n(x) = \cos(2n+1)\pi x,$$
$$\phi_n(x) = \left(\frac{\beta}{\beta-1} - x\right)\frac{\sin(2n+1)\pi x}{2(2n+1)\pi}$$

for $n = 0,1,2,\ldots$respectively. The biorthogonal systems analogous to (16), (17) are

$$\left\{\sin(2n+1)\pi x, \frac{4\left(\overline{\beta}-1\right)}{\overline{\beta}+1}\left(x + \frac{1}{\overline{\beta}-1}\right)\cos(2n+1)\pi x\right\}_{n=0}^{\infty} \tag{75}$$

$$\left\{\cos(2n+1)\pi x, \frac{4(\beta-1)}{\beta+1}\left(\frac{\beta}{\beta-1} - x\right)\sin(2n+1)\pi x\right\}_{n=0}^{\infty} \tag{76}$$

respectively.

Analogous formulas to (18) and (19) are

$$\left(\lambda_{N,j} - ((2n+1)\pi)^2\right)(\Psi_{N,j}, \sin(2n+1)\pi x) = (q(x)\Psi_{N,j}, \sin(2n+1)\pi x) \tag{77}$$

$$\left(\lambda_{N,j} - ((2n+1)\pi)^2\right)(\Psi_{N,j}, \varphi_n^*) - (2n+1)\gamma_2(\Psi_{N,j}, \sin(2n+1)\pi x) = (q(x)\Psi_{N,j}, \varphi_n^*) \tag{78}$$

respectively, where

$$\gamma_2 = \frac{8\pi(\beta-1)}{\beta+1}.$$

Instead of (16)-(19) using (75)-(78) and arguing as in the proofs of Theorem 1 and Theorem 2 we obtain the following results for T_2.

Theorem 4 *If (9) holds, then the large eigenvalues $\lambda_{n,j,2}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j,2} = ((2n+1)\pi)^2 + (-1)^j\,\frac{\sqrt{2\gamma_2}}{2}\sqrt{(2n+1)\,s_{2n+1}}(1+o(1)). \tag{79}$$

for $j=1,2$. The eigenfunctions $\Psi_{n,j,2}$ corresponding to $\lambda_{n,j,2}$ obey

$$\Psi_{n,j,2} = \sqrt{2}\cos(2n+1)\pi x + O\left(n^{-1/2}\right). \tag{80}$$

Moreover, if there exists a sequence $\{n_k\}$ such that (9) holds when n is replaced by n_k, then the root functions of T_2 do not form a Riesz basis.

Lastly we consider the operator T_4. It is well-known that (see formulas (47a), (47b)) in page 65 of [18]) the eigenvalues of the operators $T_4(q)$ consist of the sequences $\{\lambda_{n,1,4}\}, \{\lambda_{n,2,4}\}$ satisfying (74) when $\lambda_{n,j,2}$ is replaced by $\lambda_{n,j,4}$. The eigenvalues, eigenfunctions and associated functions of T_4 are

$$\lambda_n = (\pi + 2\pi n)^2\,,\ y_n(x) = \sin(2n+1)\pi x,$$
$$\phi_n(x) = \left(\frac{\alpha}{1-\alpha} + x\right)\frac{\cos(2n+1)\pi x}{2(2n+1)\pi}$$

for $n = 0, 1, 2, \ldots$respectively. The biorthogonal systems analogous to (16), (17) are

$$\left\{\cos(2n+1)\pi x, \frac{4(1-\overline{\alpha})}{1+\overline{\alpha}}\left(\frac{1}{1-\overline{\alpha}} - x\right)\sin(2n+1)\pi x\right\}_{n=0}^{\infty} \tag{81}$$

$$\left\{\sin(2n+1)\pi x, \frac{4(1-\alpha)}{1+\alpha}\left(\frac{\alpha}{1-\alpha} + x\right)\cos(2n+1)\pi x\right\}_{n=0}^{\infty} \tag{82}$$

respectively.

Analogous formulas to (18) and (19) are

$$\left(\lambda_{N,j} - (\pi + 2\pi n)^2\right)(\Psi_{N,j}, \cos(2n+1)\pi x) = (q(x)\Psi_{N,j}, \cos(2n+1)\pi x), \tag{83}$$

$$\left(\lambda_{N,j} - ((2n+1)\pi)^2\right)(\Psi_{N,j}, \varphi_n^*) - (2n+1)\gamma_4(\Psi_{N,j}, \cos(2n+1)\pi x) = (q(x)\Psi_{N,j}, \varphi_n^*) \tag{84}$$

respectively, where

$$\gamma_4 = \frac{8\pi(1-\alpha)}{1+\alpha}.$$

Instead of (16)-(19) using (81)-(84) and arguing as in the proofs of Theorem 1 and Theorem 2 we obtain the following results for T_4.

Theorem 5 *If (9) holds, then the large eigenvalues $\lambda_{n,j,4}$ are simple and satisfy the following asymptotic formulas*

$$\lambda_{n,j,4} = ((2n+1)\pi)^2 + (-1)^j \frac{\sqrt{2\gamma_4}}{2}\sqrt{(2n+1)s_{2n+1}}(1+o(1)). \tag{85}$$

for $j = 1, 2$. The eigenfunctions $\Psi_{n,j,4}$ corresponding to $\lambda_{n,j,4}$ obey

$$\Psi_{n,j,4} = \sqrt{2}\sin(2n+1)\pi x + O\left(n^{-1/2}\right). \tag{86}$$

Moreover, if there exists a sequence $\{n_k\}$ such that (9) holds when n is replaced by n_k, then the root functions of T_4 do not form a Riesz basis.

Remark 1 *Suppose that*

$$\int_0^1 xq(x)\,dx \neq 0. \tag{87}$$

If

$$\frac{1}{2}s_{2n} + B = o\left(\frac{1}{n}\right), \tag{88}$$

where B is defined by (34), then arguing as in the proof of Theorem 2, we obtain that the large eigenvalues of the operator T_1 are simple. Moreover if there exists a sequence $\{n_k\}$ such that (88) holds when n is replaced by n_k, then the root functions of T_1 do not form a Riesz basis. The similar results can be obtained for the operators T_2, T_3 and T_4.

Remark 2 *Using (31) and (35) and arguing as in the proof of Theorem 3 of [1] it can be obtained asymptotic formulas of arbitrary order for the eigenvalues and eigenfunctions of the operator T_1. The similar formulas can be obtained for the operators T_2, T_3 and T_4.*

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